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On the problem of mass dependence of the two-point function of the real scalar free massive field on the light cone

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Abstract
We investigate the generally assumed inconsistency in light cone quantum field theory that the restriction of a massive, real scalar free field to the nullplane $\Sigma = \{x^0 + x^3 = 0\}$ is independent of mass (Leutwyler, Klauder and Streit 1970 *Nuovo Cimento* A **66** 536), but the restriction of the two-point function is mass dependent (see, e.g., Nakanishi and Yamawaki 1977 *Nucl. Phys.* B **122** 15; Yamawaki K 1997 *Proc. Int. Workshop New Nonperturbative Methods and Quantization on the Light Cone (Les Houches, France) Preprint* hep-th/9707141). We resolve this inconsistency by showing that the two-point function has no canonical restriction to Σ in the sense of distribution theory. Only the so-called tame restriction of the two-point function, which we have introduced in (Ullrich P 2004 Uniqueness in the characteristic Cauchy problem of the Klein–Gordon equation and tame restrictions of generalized functions *Preprint* math-ph/0408022 (submitted)) exists. Furthermore, we show that this tame restriction is indeed independent of the mass. Hence the inconsistency is induced by the erroneous assumption that the two-point function has a (canonical) restriction to Σ.

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1. Introduction

Let $\phi(x)$ be the real scalar free quantum field of mass $m > 0$, and let $|0\rangle$ denote the (unique) vacuum state. The (Wightman) n-point functions (or vacuum expectation values) are defined by $W_n(x_1, \ldots, x_n) = \langle 0| \phi(x_1) \cdots \phi(x_n) |0\rangle$ $(n \in \mathbb{N})$. Since ϕ is a free field, the two-point

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function $W_2(x, y)$ is explicitly given by $W_2(x, y) = -\mathrm{i}D_m^{(-)}(x - y)$, where $D_m^{(-)}(x)$ is the negative frequency Pauli–Jordan function $\left(\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}\right)$

$$D_m^{(-)}(x) = \frac{-1}{\mathrm{i}(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{p}}{2\omega(\mathbf{p})}\, \mathrm{e}^{-\mathrm{i}(\omega(\mathbf{p})x^0 + \mathbf{x}\cdot\mathbf{p})}.$$

Treating the field ϕ in the framework of light cone quantization, the canonical commutator relation reads

$$[\widetilde{\phi}(\tilde{x})\widetilde{\phi}(\tilde{y})]_{x^+=y^+=0} = \frac{1}{4\mathrm{i}}\epsilon(x^- - y^-)\delta(\mathbf{x}_\perp - \mathbf{y}_\perp), \tag{1.1}$$

where ϵ is the sign function and we have introduced *light cone coordinates*

$$\tilde{x} = (x^+, \tilde{\mathbf{x}}) = (x^+, \mathbf{x}_\perp, x^-) = \kappa(x^0, x^1, x^2, x^3)$$

by

$$x^+ = (1/\sqrt{2})(x^0 + x^3), \qquad \mathbf{x}_\perp = (x^1, x^2), \qquad x^- = (1/\sqrt{2})(x^0 - x^3).$$

Furthermore, $\widetilde{\phi}(\tilde{x}) = \phi(\kappa^{-1}(\tilde{x}))$ denotes the transformed field. There is a generally alleged inconsistency in light cone quantum field theory (see, for example [10, 17]) which we explain now in detail: using the commutator relation (1.1) one formally obtains the equation

$$\langle 0|\,\widetilde{\phi}(\tilde{x})\widetilde{\phi}(0)\,|0\rangle_{x^+=0} = \frac{1}{2\pi}\int_{p^+>0} \frac{\mathrm{d}p^+}{2p^+}\, \mathrm{e}^{-\mathrm{i}p^+x^-}\delta(\mathbf{x}_\perp), \tag{1.2}$$

where the right-hand side obviously does not depend on the mass. Since $W_2(x, y) = -\mathrm{i}D_m^{(-)}(x - y)$, (1.2) should be equal to $-\mathrm{i}$ times the restriction of $\widetilde{D}_m^{(-)}(\tilde{x})$ to $x^+ = 0$, where $\widetilde{D}_m^{(-)}(\tilde{x}) = D_m^{(-)}(\kappa^{-1}(\tilde{x}))$ denotes the negative frequency Pauli–Jordan function transformed to light cone coordinates. In (3+1)-dimensional Minkowski space $\mathbb{M}^4$ $D_m^{(-)}(x)$ has the following explicit representation [1]:

$$D_m^{(-)}(x) = \lim_{\substack{\xi\to 0\\ \xi\in V^+}} \frac{\mathrm{i}m^2}{4\pi^2} h(-m^2(x - \mathrm{i}\xi)^2), \tag{1.3}$$

where $V^+ = \{p \in \mathbb{M}^4 : p^2 > 0, p^0 > 0\}$, $h(\zeta) = K_1(\sqrt{\zeta})/\sqrt{\zeta}$, K_1 is the modified Bessel function of second kind and the branch of $\sqrt{\zeta}$ is taken to be positive for $\zeta > 0$. One seemingly obtains a contradiction by transforming formally the right-hand side of (1.3) to LC-coordinates and putting $x^+ = 0$, because then the right-hand side remains dependent on the mass m. However, as we will see later, the formal manipulations at the right-hand side of (1.3) are ill-defined, since $D_m^{(-)}(x)$ has no (canonical) restriction to $\{x^0 + x^3 = 0\}$. More precisely, the operations of taking the limit $\xi \to 0$ $(\xi \in V^+)$ (in $\mathcal{S}'(\mathbb{R}^4)$—the space of tempered distributions) and putting $x^+ = 0$ do not commute in (1.3).

2. Notations and conventions

Already in the introduction we have introduced light cone coordinates $\tilde{x} = \kappa(x)$ by using the Kogut–Soper convention [2], where $x = (x^\mu)$ are Minkowski coordinates. As usual in light cone physics one writes

$$\tilde{x} = (x^+, \tilde{\mathbf{x}}) = (x^+, \mathbf{x}_\perp, x^-), \qquad \mathbf{x}_\perp = (x^1, x^2).$$

The Minkowski bilinear form $\langle x, y\rangle_{\mathbb{M}} = x^\mu x_\mu = x^\mu g_{\mu\nu}x^\nu$, where $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ is the usual Minkowski metric, transforms to the so-called LC-bilinear from

$$\langle\tilde{x}, \tilde{y}\rangle_{\mathbb{L}} = \langle\kappa^{-1}(\tilde{x}), \kappa^{-1}(\tilde{y})\rangle_{\mathbb{M}} = x^+y^- + x^-y^+ - \mathbf{x}_\perp\cdot\mathbf{y}_\perp \qquad (\mathbf{x}_\perp\cdot\mathbf{y}_\perp = x^1y^1 + x^2y^2)$$

when going over from Minkowski coordinates to light cone coordinates; hereby $\kappa : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear transformation from Minkowski coordinates to light cone coordinates.

We also use light cone coordinates $\tilde{p} = \kappa(p)$ in momentum space. However, since x^+ is the time variable in light cone physics and x^+ is multiplied by p^- in $\langle x, p\rangle_{\mathbb{L}}$, the variable p^- takes on the role of energy and $\tilde{\mathbf{p}} = (p^+, \mathbf{p}_\perp)$ is the (light cone) spatial momentum. Hence the light cone variable $\tilde{p}$ is split into $\tilde{p} = (\tilde{\mathbf{p}}, p^-)$ with $\tilde{\mathbf{p}} = (p^+, \mathbf{p}_\perp)$ in contrast to the LC-spacetime variable $\tilde{x}$ which we have split into $\tilde{x} = (x^+, \tilde{\mathbf{x}})$ with $\tilde{\mathbf{x}} = (\mathbf{x}_\perp, x^-)$. Here a little bit care is needed.

Throughout this paper we denote by $\Sigma_\tau (\tau \in \mathbb{R})$ the linear subspace

$$\Sigma_\tau = \{x \in \mathbb{R}^4 : (1/\sqrt{2})(x^0 + x^3) = \tau\},$$

where, especially for $\tau = 0$, we set $\Sigma = \Sigma_0$. Note that in light cone coordinates Σ_τ reads $\{x^+ = \tau\}$, i.e., $\kappa(\Sigma_\tau) = \{\tilde{x} \in \mathbb{R}^4 : x^+ = \tau\}$.

If $U \subset \mathbb{R}^m$ is an open set, we denote by $\mathcal{D}(U)$ the (complex) vector space consisting of all (complex-valued) smooth, i.e., C^∞ functions on U with compact support. On $\mathcal{D}(U)$ one defines a topology which makes $\mathcal{D}(U)$ into a complete locally convex space [8, 12], the dual space $\mathcal{D}'(U)$, i.e., the vector space of all linear, continuous functionals on $\mathcal{D}(U)$ is called the space of *distributions*. If $\phi \in \mathcal{D}'(U)$ is a distribution and $f \in \mathcal{D}(U)$ a test function, we denote by $\phi(f)$ as well as by (ϕ, f) the evaluation of ϕ at f. It is well known [12] that every locally integrable, complex-valued function φ on U induces a distribution by $\Lambda_\varphi(f) = \int \varphi f \, \mathrm{d}x$, and that φ is uniquely determined by its induced distribution almost everywhere. This especially implies that the mapping, which maps a continuous function to its induced distribution, is one-to-one and hence, by identifying a function with its induced distribution, we can write $C^0(U) \subset \mathcal{D}'(U)$; we also have $\mathcal{D}(U) \subset C^k(U) \subset \mathcal{D}'(U)$ for all $0 \leqslant k \leqslant \infty$. On the space $\mathcal{D}'(U)$ of distributions one usually installs the so-called weak*-topology [12]. An essential feature of distribution theory is the fact that with respect to the weak*-topology $\mathcal{D}(U)$ is a dense subspace of $\mathcal{D}'(U)$; it holds even that for every distribution $\varphi \in \mathcal{D}'(U)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in \mathcal{D}(U)$ which converges to φ in $\mathcal{D}'(U)$. Thus any sequentially continuous function on $\mathcal{D}'(U)$ is uniquely determined by its values on $\mathcal{D}(U)$.

Along with $\mathcal{D}(\mathbb{R}^m)$ one introduces the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ of rapidly decreasing functions and defines on $\mathcal{S}(\mathbb{R}^m)$ a topology which makes $\mathcal{S}(\mathbb{R}^m)$ into a Fréchet space. The dual space $\mathcal{S}'(\mathbb{R}^n)$ is called the space of *tempered distributions (or generalized functions)* [1, 8, 12]. As in the case of distributions we may assume $\mathcal{S}(\mathbb{R}^m) \subset \mathcal{S}'(\mathbb{R}^m)$. Furthermore $\mathcal{S}(\mathbb{R}^m)$ is dense in $\mathcal{S}'(\mathbb{R}^m)$ where $\mathcal{S}'(\mathbb{R}^m)$ is endowed with the weak*-topology. Note that $\mathcal{D}(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m)$, but the topology of $\mathcal{D}(\mathbb{R}^m)$ is finer than the subspace topology induced by $\mathcal{S}(R^m)$. One usually identifies $\mathcal{S}'(\mathbb{R}^m)$ with the subspace of distributions ($\in \mathcal{D}'(\mathbb{R}^m)$) which admit a linear, continuous extension to $\mathcal{S}(\mathbb{R}^m)$.

3. The canonical restriction of a distribution

In this section we summarize some well known results from distribution theory [8] which will be needed in the sequel.

As already mentioned in the previous section, one has a chain of inclusions $\mathcal{D}(U) \subset C^k(U) \subset \mathcal{D}'(U) (0 \leqslant k \leqslant \infty)$. Since distributions can be approximated by smooth functions in $\mathcal{D}(U)$ with respect to the weak*-topology of $\mathcal{D}'(U)$, it is possible to define operations on distributions by continuous extension from the smooth case. However, there are operations on functions which cannot be extended to the whole of $\mathcal{D}'(U)$ due to the presence of singularities. Thus one has to remain in appropriate subspaces of $\mathcal{D}'(U)$ by taking care of the singularities of distributions. Prominently, the definition of the canonical product of distributions causes problems with far reaching consequences in quantum field theory. The product operation of distributions is strongly related to the restriction operation which is a special case of the so-called pullback operation. We explain the pullback operation and its connection to the (canonical) restriction of distributions in appendix B.

Assume $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are non-empty, open subsets and $a \in U$ is fixed. One would like to define the restriction $\phi|_{x=a} = \phi(a, y)$ of a distribution $\phi \in \mathcal{D}'(U \times V)$ in a canonical way, i.e., by continuous extension from the smooth case. To see, how this can be done, assume ϕ is a smooth function, i.e., $\phi(x, y) \in \mathcal{D}(U \times V)$. Then, by Fourier's inversion formula, for all $f(y) \in \mathcal{D}(V)$

$$(\phi(a, y), f(y)) = (2\pi)^{-(n+m)} \int \mathrm{d}^m p\, \mathrm{d}^n q\, \hat{\phi}(p, q)\, \mathrm{e}^{\mathrm{i}\langle a,p\rangle} \hat{f}(-q), \tag{3.1}$$

where '$\hat{}$' means taking the Fourier transform. The right-hand side of (3.1) now serves as the defining formula for the canonical restriction of a distribution $\phi(x, y) \in \mathcal{D}'(U \times V)$ to the hyperplane $\{x = a\}$ [8]. Note that if the distribution $\phi(x, y)$ has compact support the Fourier transform $\hat{\phi}(p, q)$ is a smooth (even entire analytic) function by a general theorem of Paley and Wiener (see, e.g., [12]). If $\phi(x, y)$ has not necessarily compact support one easily reduces to the case of compact support by localization [8]. Note that distributions can be glued together such that it is only necessary to define the restriction locally. The right-hand side of (3.1) is not well-defined for any distribution $\phi \in \mathcal{D}'(U \times V)$ (with compact support) as can easily be seen by the following example: let $\phi(x, y)$ be the delta function $\delta(x, y)$. As is well known (e.g. [12]), the Fourier transform of $\delta(x, y)$ is the constant function 1, i.e. $\hat{\delta}(p, q) = 1$, and thus the integral in (3.1) is not convergent. To obtain a well-defined distribution by the right-hand side of (3.1) one has to put conditions on the Fourier transform $\hat{\phi}(p, q)$. Since $\hat{f}(q)$ is rapidly decreasing at infinity, one only needs conditions for the asymptotic behaviour of $\hat{\phi}(p, q)$ in a region of the form $|p| > \epsilon|q|$ for some $\epsilon > 0$; note that this is an open conic neighbourhood for all points of the form $(p, 0) p \in \mathbb{R}^m \backslash 0$. At this point the so-called wave front set [8, 11] enters the scene. For convenience of the reader we discuss the wave front set of a distribution in more detail in appendix A. For the moment, however, it is enough to know that the wave front set of a distribution $\phi \in \mathcal{D}'(X) (X \subset \mathbb{R}^n$ open) is a subset $\mathrm{WF}(\phi) \subset X \times \mathbb{R}^n \backslash 0$ which encodes the behaviour at infinity of the Fourier transform of (localizations of) ϕ. In view of (3.1) this information is crucial in defining the restriction of a distribution. From theorem appendix B.1 [4] one gets the following criterion which also serves as the definition of the (canonical) restriction of a distribution.

Proposition 3.1. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets and $a \in U$. Then the restriction $\phi|_{x=a}$ can be defined in one and only one way for all $\phi(x, y) \in \mathcal{D}'(U \times V)$ with*

$$\{((a, y), (p, 0)) \mid y \in V, p \in \mathbb{R}^m\} \cap WF(\phi) = \emptyset \tag{3.2}$$

so that $\phi|_{x=a} = \phi(a, \cdot)$ when $\phi \in C^\infty(U \times V)$.

Definition 3.2. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets and $a \in U$. Then we say that $\phi(x, y) \in \mathcal{D}'(U \times V)$ has a canonical restriction to $\{x = a\}$ if (3.2) holds. The distribution $\phi(a, y) = \phi|_{x=a}$ given by proposition 3.1 is called the canonical restriction of ϕ to $\{x = a\}$.*

Remark 3.3. (a) If $\phi(x, y) \in \mathcal{D}'(U \times V)$ has a restriction to $\{x = a\}$ then also any $\partial_x^\alpha \partial_y^\beta \phi(x, y) (\alpha, \beta$ multi-indices) has a restriction to $\{x = a\}$ – by [8], $\mathrm{WF}(\partial_x^\alpha \partial_y^\beta \phi) \subset \mathrm{WF}(\phi)$.

(b) Since the wave front set is a closed set one easily finds that if $\phi(x, y) \in \mathcal{D}'(U \times V)$ has a restriction to $\{x = a\}$ then there is an open neighbourhood $U' \subset U$ of a such that $\phi(x, y)$ has a restriction to $\{x = a'\}$ for all $a' \in U'$.

[4] See the discussion after corollary appendix B.2.

(c) Condition (3.2) in the definition of the restriction of a distribution looks a little bit artificial. However, one can show (see, e.g. [16]) that $\phi(x, y) \in \mathcal{D}'(U \times V)$ has a restriction to $\{x = a\}$ if and only if, sufficiently close to $x = a$, $\phi(x, y)$ is C^∞-dependent on x as a parameter; i.e., there is an open neighbourhood $W \subset U$ of a and a family $(\phi_x)_{x\in W}$ in $\mathcal{D}'(V)$ such that $W \to \mathbb{C}, x \mapsto (\phi_x, g)$ is C^∞ for every $g \in \mathcal{D}(V)$ and

$$(\phi(x, y), f(x)g(y)) = \int_W (\phi_x, g) f(x)\,\mathrm{d}x \tag{3.3}$$

for all $f(x) \in \mathcal{D}(W)$, $g(y) \in \mathcal{D}(V)$; if this is the case $\phi(a, y) = \phi_a(y)$. Note that the family $(\phi_x)_{x\in W}$ is uniquely determined by (3.3). Clearly, defining the restriction of a distribution using the notion of parameter dependence (see, e.g. [1]) would be more intuitive. However, this approach is more appropriate if one wants to show that the restriction of a distribution does exist since one simply has to verify (3.3)—see the following example of the Pauli–Jordan function. In the next section we would like to prove nonexistence for which, in the opinion of the authors, the notion of parameter dependence is improper. The more technical notion of wave front set provides the right tool to prove nonexistence of restrictions.

Example 3.4. The Pauli–Jordan function

$$D_m(x) = \frac{1}{\mathrm{i}(2\pi)^3}\int \mathrm{d}^4p\,\epsilon(p^0)\delta(p^2 - m^2)\,\mathrm{e}^{\mathrm{i}\langle p,x\rangle_{\mathbb{M}}} \in \mathcal{S}'(\mathbb{R}^4)$$

has a canonical restriction to $\{x^0 = 0\}$. Moreover, $D_m(x)$ is a fundamental solution of the Klein–Gordon operator, i.e.,

$$D_m(0, \mathbf{x}) = 0, \qquad (\partial_{x^0} D_m)(0, \mathbf{x}) = \delta(\mathbf{x}).$$

That $D_m(x)$ has a restriction to $\{x^0 = 0\}$ can either be seen by considering the wave front set of D_m or, more explicitly, by showing that $D_m(x^0, \mathbf{x})$ is C^∞-dependent on x^0 as a parameter[5]:

$$(D_m)_{x^0}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{\omega(\mathbf{p})}\sin(\omega(\mathbf{p})x^0)\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}.$$

Also the positive- and negative-frequency parts $D_m^{(\pm)}(x)$ have restrictions to $\{x^0 = \tau\}(\tau \in \mathbb{R})$. In [11] the wave front set of $D_m^{(-)}(x)$ is explicitly determined:

$$\mathrm{WF}\big(D_m^{(-)}(x)\big) = W_0^{(-)} \cup W_+^{(-)} \cup W_-^{(-)}, \qquad \text{with}$$
$$W_0^{(-)} = \{(0, p) : p \in \Gamma^-\backslash 0\},\ W_\pm^{(-)} = \{(p, \mp\lambda p) : p \in \Gamma^\pm\backslash 0, \lambda > 0)\},$$

where $\Gamma^\pm = \{p \in \mathbb{M}^4 : p^2 = 0, \pm p^0 > 0\}$. Thus, condition (3.2) holds for $D_m^{(-)}$. Since $D_m^{(+)}(x) = -D_m^{(-)}(-x)$ we have $\mathrm{WF}\big(D_m^{(+)}\big) = -\mathrm{WF}\big(D_m^{(-)}\big)$ and hence the same holds also for $D_m^{(+)}$.

In figure 1 we have illustrated the wave front set of $D_m^{(-)}$. Each element (x, p) of $\mathrm{WF}\big(D_m^{(-)}\big)$ is represented by a pointed vector with base point x and unit vector in the direction of p.

So far we have only defined the restriction of a distribution $\phi(x, y)$ to a hyperplane of the form $\{x = a\}(a \in \mathbb{R}^m)$. However, any smooth submanifold of $\mathbb{R}^m$ can be described locally in such a manner using appropriate charts. Let $\Sigma_\tau = \{(1/\sqrt{2})(x^0 + x^3) = \tau\}(\tau \in \mathbb{R})$ and κ the linear transformation to light cone coordinates, then $\Sigma_\tau = \{\kappa^{-1}(\tilde{x}) : x^+ = \tau\}$.

[5] This is indeed true for every solution of the Klein–Gordon equation since the Klein–Gordon operator is hypoelliptic with respect to x^0, see [4–6].

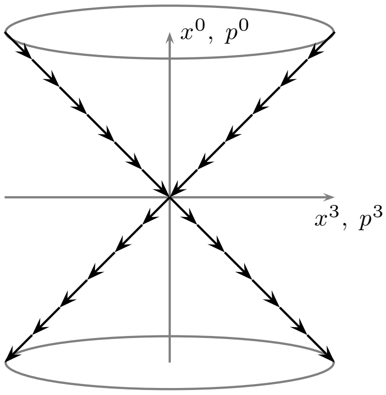

Figure 1. The wave front set of $D_m^{(-)}$.

Definition 3.5. *A distribution $\phi(x) \in \mathcal{D}'(\mathbb{R}^4)$ has a (canonical) restriction to $\Sigma_\tau (\tau \in \mathbb{R})$ if $\kappa_*\phi(\tilde{x}) = (\phi \circ \kappa^{-1})(\tilde{x})$ has a (canonical) restriction to $\{x^+ = \tau\}$. In this case we call $\phi|_{\Sigma_\tau} = \kappa_*\phi(\tau, \tilde{\mathbf{x}}) \in \mathcal{D}'(\mathbb{R}^3)$ the (canonical) restriction of ϕ to Σ_τ.*

Remark 3.6. More generally, one can define the restriction of a distribution $\phi(x_1, \dots, x_r) \in \mathcal{D}'(\mathbb{R}^4 \times \cdots \times \mathbb{R}^4)$ to $\Sigma_{\tau_1} \times \cdots \times \Sigma_{\tau_r}$ as the restriction of $\phi(\kappa^{-1}(\tilde{x}_1), \dots, \kappa^{-1}(\tilde{x}_r))$ to $\{x_1^+ = \tau_1, \dots, x_r^+ = \tau_r\}$, where $\tilde{x}_i = (x_i^+, \tilde{\mathbf{x}}_i)$ and $\tau_i \in \mathbb{R}(i = 1, \dots, r)$.

4. Nonexistence of the restriction of the two-point function to the nullplane

Since we have explicit knowledge of the wave front set of $D_m^{(-)}$ it is easy now to show that the two-point function $W_2(x, y)$ has no (canonical) restriction to $\Sigma \times \Sigma$.

Theorem 4.1. *Let $W_2(x, y) \in \mathcal{D}'(\mathbb{R}^4 \times \mathbb{R}^4)$ denote the two-point function of the real scalar free massive field. Then $W_2(x, y)$ has no canonical restriction to $\Sigma \times \Sigma = \{x^0 + x^3 = y^0 + y^3 = 0\}$.*

Proof. Since $W_2(x, y) = -\mathrm{i}D_m^{(-)}(x - y)$ it is enough to show that $D_m^{(-)}(x)$ has no canonical restriction to $\Sigma = \{x^0 + x^3 = 0\}$. By remark appendix B.3 we have to show that $N_\lambda \cap \mathrm{WF}\big(D_m^{(-)}\big) \neq \emptyset$ where N_λ is the set of normals of $\lambda : \mathbb{R}^3 \to \mathbb{R}^4, (x^1, x^2, x^-) \mapsto (x^-/\sqrt{2}, x^1, x^2, -x^-/\sqrt{2})$. One easily verifies that

$$N_\lambda = \{(x, p) \in \Sigma \times \mathbb{R}^4 : p^0 = p^3\}$$

and hence $N_\lambda \cap \mathrm{WF}\big(D_m^{(-)}\big) \neq \emptyset$, which is easily seen by considering figures 1 and 2. □

So far we have shown that $D_m^{(+)}(x)$ and $D_m^{(-)}(x)$ have no canonical restriction to $\Sigma = \{x^0 + x^3 = 0\}$—note that $\mathrm{WF}\big(D_m^{(+)}\big) = -\mathrm{WF}\big(D_m^{(-)}\big)$. In supplementary we will show that this also holds true for the Pauli–Jordan function D_m which is the sum of $D_m^{(+)}$ and $D_m^{(-)}$.

Proposition 4.2. *The Pauli–Jordan function $D_m(x)$ has no canonical restriction to Σ.*

Proof. We will show that $\mathrm{WF}(D_m) = \mathrm{WF}\big(D_m^{(+)}\big) \cup \mathrm{WF}\big(D_m^{(-)}\big)$; the assertion follows then from the proof of theorem 4.1. Since $D_m = D_m^{(+)} + D_m^{(-)}$ we get only one direction, namely $\mathrm{WF}(D_m) \subset \mathrm{WF}\big(D_m^{(+)}\big) \cup \mathrm{WF}\big(D_m^{(-)}\big)$. To prove the other inclusion we may assume w.l.o.g. that $\mathrm{WF}(D_m) \cap \mathrm{WF}\big(D_m^{(-)}\big) \neq \emptyset$. Since $\mathcal{L}_+^\uparrow$, the restricted Lorentz group,

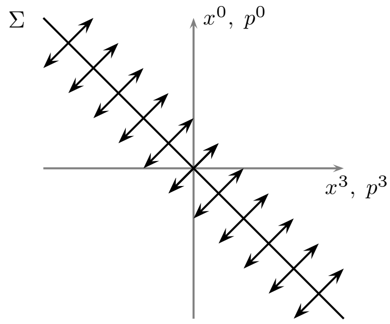

Figure 2. The set of normals N_λ.

operates transitively on $\mathrm{WF}\big(D_m^{(-)}\big)$ and since D_m, and hence $\mathrm{WF}(D_m)$, is invariant under $\mathcal{L}_+^\uparrow$, one obtains $\mathrm{WF}\big(D_m^{(-)}\big) \subset \mathrm{WF}(D_m)$ (see also the proof of theorem IX.48 in [11]). Furthermore, $\mathrm{WF}\big(D_m^{(+)}\big) \subset \mathrm{WF}(D_m)$ since $\mathrm{WF}(D_m^{(+)}(x)) = -\mathrm{WF}(D_m^{(-)}(x)) \subset -\mathrm{WF}(D_m)$ and $-\mathrm{WF}(D_m(x)) = \mathrm{WF}(D_m(-x)) = \mathrm{WF}(D_m(x))$. □

5. The tame restriction of the two-point function

The nonexistence of the restriction of the Pauli–Jordan function to $\Sigma = \{x^0 + x^3 = 0\}$ is related to a fundamental problem in light cone quantum field theory where one describes the dynamics of a quantum field by using $x^+ = (1/\sqrt{2})(x^0 + x^3)$ as 'time'-evolution parameter. In this context it is essential to have well-defined fields for fixed $x^+ = \text{const}$. However, to carry out the standard construction of a free field for fixed time, one has to remain in a proper subspace of $\mathcal{S}(\mathbb{R}^3)$ [9] which was considered as a fault of the theory [13]. In [14] this problem was solved by introducing a new test function space $\mathcal{S}_{\partial_-}(\mathbb{R}^3)$ on which the 'restriction' of the free field can be defined and which determines the covariant field uniquely—we call this the 'tame restriction' of the free field to Σ. Now, since the covariant commutator relation of a free field ϕ reads

$$[\phi(x), \phi(y)] = -\mathrm{i}D_m(x-y), \tag{5.1}$$

where D_m is the Pauli–Jordan function, we see that the problem of nonexistence of the real scalar field on Σ results in the nonexistence of the restriction of the Pauli–Jordan function to Σ. In [15] we introduced the tame restriction of a generalized function and computed it for the Pauli–Jordan function, where we obtained $(1/4)\delta(\mathbf{x}_\perp) \otimes \epsilon(x^-)$. Hence, if we take the tame restrictions (to Σ) on both sides of (5.1) we arrive at the well-known commutator relation of light cone quantum field theory [3]. The same happens with the two-point function $W_2(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle$ since

$$\langle 0|\phi(x)\phi(y)|0\rangle = -\mathrm{i}D_m^{(-)}(x-y), \tag{5.2}$$

and $D_m^{(-)}$ does not have a canonical restriction to Σ. However, since $D_m^{(-)}$ is a solution of the Klein–Gordon equation $(\Box + m^2)D_m^{(-)} = 0$ we know from [15] that $D_m^{(-)}$ admits a tame restriction to Σ. In the sequel we will compute this tame restriction explicitly and show that it is independent of the mass. Since the tame restriction of the free field to Σ is also independent of the mass [14, 16] no inconsistency appears if we take the tame restrictions (to Σ) on both sides of (5.2). First of all we have to recall the definition of the tame restriction of a generalized function to Σ—for details see [14, 15].

Definition 5.1. *(a) Let* $\mathcal{S}_{p^+}(\mathbb{R}^n) = \bigcap_{k\in\mathbb{N}_0}\{(p^+)^k g : g \in \mathcal{S}(\mathbb{R}^n)\}$ *be the topological vector space endowed with the subspace topology induced by* $\mathcal{S}(\mathbb{R}^n)$*; the dual space* $\mathcal{S}'_{p^+}(\mathbb{R}^n)$ *is called the space of squeezed generalized functions.*

(b) Let $\mathcal{S}_{\partial_-}(\mathbb{R}^n) = \bigcap_{k\in\mathbb{N}_0}\{\partial^k_{x^-} g : g \in \mathcal{S}(\mathbb{R}^n)\}$ *be the topological vector space endowed with the subspace topology induced by* $\mathcal{S}(\mathbb{R}^n)$*; the dual space* $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ *is called the space of tame generalized functions.*

Remark 5.2. The achievement of definition 5.1 (a) is the following: any function $h \in \mathcal{S}_{p^+}(\mathbb{R}^n)$ (as well as any of its derivatives) goes to zero for $|p^+| \to \infty$ due to the rapidly decreasing behaviour of $g \in \mathcal{S}(\mathbb{R}^n)$ and it also goes to zero for $|p^+| \to 0$ faster than any power of $|p^+|$ due to the presence of the factor $(p^+)^k$ for all $k \in \mathbb{N}_0$. Definition 5.1 (b) is induced from (a) via the Fourier transform. Recall that the Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$ and maps $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ onto $\mathcal{S}_{p^+}(\mathbb{R}^n)$. Furthermore, the spaces $\mathcal{S}_{p^+}(\mathbb{R}^n)$ and $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ are Fréchet spaces.

Since we are using light cone coordinates, we also have to introduce the so-called $\mathbb{L}$-Fourier transformation $\mathcal{F}_{\mathbb{L}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defined by

$$\mathcal{F}_{\mathbb{L}}(f)(\tilde{p}) = \int f(\tilde{x})\, \mathrm{e}^{\mathrm{i}\langle\tilde{x},\tilde{p}\rangle_{\mathbb{L}}}\, \mathrm{d}\tilde{x},$$

where $\langle\tilde{x},\tilde{p}\rangle_{\mathbb{L}} = x^+p^- + x^-p^+ - \mathbf{x}_\perp\cdot\mathbf{p}_\perp$. Since x^+ is the time variable in light cone physics we also introduce the spatial part of the $\mathbb{L}$-Fourier transformation

$$f^{\sqcap}(\tilde{\mathbf{p}}) = \mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(f)(\tilde{\mathbf{p}}) = \int f(\tilde{\mathbf{x}})\, \mathrm{e}^{\mathrm{i}(x^-p^+ - \mathbf{x}_\perp\cdot\mathbf{p}_\perp)}\, \mathrm{d}\tilde{\mathbf{x}},$$

which, in the special case of only one dimension, reads

$$f^{\sqcap}(p^+) = \mathcal{F}_{\mathbb{L}}^{x^-\to p^+}(f)(p^+) = \int f(x^-)\, \mathrm{e}^{\mathrm{i}x^-p^+}\, \mathrm{d}x^-.$$

Clearly, $\mathcal{F}_{\mathbb{L}}$, $\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}$ and $\mathcal{F}_{\mathbb{L}}^{x^-\to p^+}$ are isomorphisms from $\mathcal{S}(\mathbb{R}^q)$ onto $\mathcal{S}(\mathbb{R}^q)$ which map $\mathcal{S}_{\partial_-}(\mathbb{R}^q)$ onto $\mathcal{S}_{p^+}(\mathbb{R}^q)$ (q appropriately chosen) and which extend canonically to sequentially continuous maps from $\mathcal{S}'(\mathbb{R}^q)$ onto $\mathcal{S}'(\mathbb{R}^q)$ respectively from $\mathcal{S}'_{\partial_-}(\mathbb{R}^q)$ onto $\mathcal{S}'_{p^+}(\mathbb{R}^q)$.

Definition 5.3 (tame restriction)**.** *(a) A generalized function* $\phi(y,z,x^-) \in \mathcal{S}'(\mathbb{R}^{m+n+1})$ *admits a tame restriction to* $\{y = y_0\}(y_0 \in \mathbb{R}^m)$ *if there is an open neighbourhood* $\Omega \subset \mathbb{R}^m$ *of* y_0 *and a family* $(\phi_y)_{y\in\Omega}$ *with* $\phi_y \in \mathcal{S}'_{\partial_-}(\mathbb{R}^{n+1})(y\in\Omega)$ *such that* $\Omega \to \mathbb{C}$, $y \mapsto (\phi_y, g)$ *is* C^∞ *for all* $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^{n+1})$ *and*

$$(\phi(y,z,x^-), f(y)g(z,x^-)) = \int_\Omega \mathrm{d}y(\phi_y, g) f(y)$$

for all $f(y) \in \mathcal{D}(\Omega)$ *and* $g(z,x^-) \in \mathcal{S}_{\partial_-}(\mathbb{R}^{n+1})$*. In this case we call* $\phi|^*_{y=y_0} = \phi_{y_0} \in \mathcal{S}'_{\partial_-}(\mathbb{R}^{n+1})$ *the tame restriction of* ϕ *to* $\{y = y_0\}$*.*

(b) A generalized function $\phi(x_1,\dots,x_r) \in \mathcal{S}'(\mathbb{R}^{4r})$ *admits a tame restriction to* $\Sigma_{\tau_1} \times \cdots \times \Sigma_{\tau_r}$ $(\Sigma_{\tau_i} = \{x_i \in \mathbb{R}^4 : (1/\sqrt{2})(x_i^0 + x_i^3) = \tau_i\}, i = 1,\dots,r)$ *if* $\phi(\kappa^{-1}(\tilde{x}_1),\dots,\kappa^{-1}(\tilde{x}_r))$ *admits a tame restriction to* $\{x_1^+ = \tau_1,\dots,x_r^+ = \tau_r\}$*; in this case we call* $\phi|^*_{\Sigma_{\tau_1}\times\cdots\times\Sigma_{\tau_r}} = \phi(\kappa^{-1}(\tilde{x}_1),\dots,\kappa^{-1}(\tilde{x}_r))|^*_{x_1^+=\tau_1,\dots,x_r^+=\tau_r}$ *the tame restriction of* $\phi(x_1,\dots,x_r)$ *to* $\Sigma_{\tau_1}\times\cdots\times\Sigma_{\tau_r}$*.*

Proposition 5.4. *Let* $D_m^{(-)}(x) \in \mathcal{S}'(\mathbb{R}^4)$ *denote the negative-frequency Pauli–Jordan function. Then* $D_m^{(-)}(x)$ *admits a tame restriction to* $\Sigma_\tau(\tau\in\mathbb{R})$ *and*[6]

$$\left(D_m^{(-)}\big|^*_{\Sigma_\tau}, g\right) = \frac{-1}{\mathrm{i}(2\pi)^3}\int_{p^+<0}\frac{\mathrm{d}^3\tilde{\mathbf{p}}}{2|p^+|}\left(\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}} g\right)(\tilde{\mathbf{p}})\, \mathrm{e}^{\mathrm{i}\tilde{\omega}(\tilde{\mathbf{p}})\tau}.$$

[6] Obviously, by $p^+ \mapsto -p^+$, the region of integration in the right-hand side can be chosen to be $\{p^+ > 0\}$.

for all $g(\tilde{\mathbf{x}}) \in \mathcal{S}_{\partial_-}(\mathbb{R}^3)$*, where* $\tilde{\omega}(\tilde{\mathbf{p}}) = (1/2p^+)\big(\mathbf{p}_\perp^2 + m^2\big)$*.*

Proof. Let $f(x^+) \in \mathcal{S}(\mathbb{R})$ and $g(\tilde{\mathbf{x}}) \in \mathcal{S}_{\partial_-}(\mathbb{R}^3)$. By definition

$$\big(\big(D_m^{(-)} \circ \kappa^{-1}\big)(x^+, \tilde{\mathbf{x}}), f(x^+)g(\tilde{\mathbf{x}})\big) = \frac{-1}{\mathrm{i}(2\pi)^3}(\delta_-(\tilde{p}^2 - m^2), \hat{f}(-p^-)g^{\sqcap}(\tilde{\mathbf{p}}))$$
$$= \frac{-1}{\mathrm{i}(2\pi)^3}\int_{p^+<0} \frac{\mathrm{d}^3\tilde{\mathbf{p}}}{2|p^+|}\hat{f}(-\tilde{\omega}(\tilde{\mathbf{p}}))g^{\sqcap}(\tilde{\mathbf{p}}). \tag{5.3}$$

Since $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^3)$ we have $f(x^+)g^{\sqcap}(\tilde{\mathbf{p}}) \in \mathcal{L}^1\big(\mathbb{R} \times \mathbb{R}^3, \mathrm{d}x^+ \otimes \frac{\mathrm{d}^n\tilde{\mathbf{p}}}{|p^+|}\big)$. Hence we can put $\hat{f}(-\tilde{\omega}(\tilde{\mathbf{p}})) = \int \mathrm{d}x^+ f(x^+)\,\mathrm{e}^{\mathrm{i}\tilde{\omega}(\tilde{\mathbf{p}})x^+}$ in (5.3), and obtain

$$\big(\big(D_m^{(-)} \circ \kappa^{-1}\big)_{x^+}, g\big) = \frac{-1}{\mathrm{i}(2\pi)^3}\int_{p^+<0} \frac{\mathrm{d}^3\tilde{\mathbf{p}}}{2|p^+|}g^{\sqcap}(\tilde{\mathbf{p}})\,\mathrm{e}^{\mathrm{i}\tilde{\omega}(\tilde{\mathbf{p}})x^+}.$$

Thus, the assertion follows since $(D_m^{(-)}|^*_{\Sigma_\tau}, g) = \big(\big(D_m^{(-)} \circ \kappa^{-1}\big)_\tau, g\big)$. □

Remark 5.5. Note that $D_m^{(-)}$ is uniquely determined by its tame restriction to Σ_0 [15].

Remark 5.6. One can easily verify that if a generalized function $\psi(x, y) \in \mathcal{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$ is of the form $\psi(x, y) = \phi(x - y)$, where $\phi \in \mathcal{S}'(\mathbb{R}^4)$, and ϕ has a tame restriction to $\tau_1 - \tau_2$ then ψ has a tame restriction to $\Sigma_{\tau_1} \times \Sigma_{\tau_2}$ and $\psi|^*_{\Sigma_{\tau_1}\times\Sigma_{\tau_r}} = \phi|^*_{\Sigma_{\tau_1-\tau_2}}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})$. Note that $(\phi(x - y), f(x)g(y)) = (\phi, f * g^\vee)$, where '*' means convolution and $g^\vee(x) = g(-x)$.

Corollary 5.7. *Let* $\phi(x)$ *be the real scalar free field of mass* $m > 0$*, and* $W_2(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle$ *the associated two-point function. Then* $W_2(x, y)$ *admits a tame restriction to* $\Sigma_\tau \times \Sigma_\tau = \{x^+ = y^+ = \tau\}(\tau \in \mathbb{R})$ *and*

$$W_2(x, y)|^*_{\Sigma_\tau\times\Sigma_\tau} = \delta(\mathbf{x}_\perp - \mathbf{y}_\perp) \otimes G(x^- - y^-) \in \mathcal{S}'_{\partial_-}(\mathbb{R}^3 \times \mathbb{R}^3),$$

where $G = \big(\mathcal{F}_{\mathbb{L}}^{x^-\to p^+}\big)^{-1}(\Theta(p^+)/p^+) \in \mathcal{S}'_{\partial_-}(\mathbb{R})$*. In particular, the tame restriction of* $W_2(x, y)$ *to* $\Sigma_\tau \times \Sigma_\tau$ *is independent of the mass.*

Proof. Since $W_2(x, y) = -\mathrm{i}D_m^{(-)}(x - y)$ it is enough to show that $D_m^{(-)}$ admits a tame restriction to $\Sigma = \{x^+ = 0\}$ and that $D_m^{(-)}|^*_\Sigma = \mathrm{i}\delta(\mathbf{x}_\perp) \otimes G(x^-)$ (cf remark 5.6); however, this follows immediately from proposition 5.4. □

6. Conclusion

To get rid of the (perturbative) zero mode and restriction problem in light cone quantum field theory, we have introduced in [14] the function space $\mathcal{S}_{\partial_-}(\mathbb{R}^3)$ and its dual space—the space of tame generalized functions. The problem that the restriction of the real scalar free massive field to $\Sigma = \{x^0 + x^3 = 0\}$ cannot be defined in the canonical way, manifests itself in the problem that the (positive-/ negative-frequency) Pauli–Jordan function has no canonical restriction to Σ in the sense of distribution theory. We were able to show the nonexistence of the restriction of the Pauli–Jordan function to Σ by considering its wave front set. Note that this is much stronger than merely showing that the common techniques cannot be applied. Thus the assumed inconsistency that the restriction of a massive, real scalar, free field to Σ is independent of the mass, but the restriction of the two-point function is mass dependent comes to nothing since *a priori* the (canonical) restriction (in the sense of distribution theory) of the two-point function does not exist. In computing the so-called tame restriction of the two-point function we obtain a (massless) result which is consistent with light cone quantization. Note that by results from [15] each solution in $\mathcal{S}'(\mathbb{R}^4)$ of the Klein–Gordon equation is uniquely determined by its tame restriction to Σ. Concluding we remark that it should be possible to develop axiomatic light cone quantum field theory on the same footing as axiomatic quantum field theory in Minkowski space by working with the unorthodox, but better adapted function space $\mathcal{S}_{\partial_-}(\mathbb{R}^3)$ instead of $\mathcal{S}(\mathbb{R}^3)$. In the case of a free field this has been done in [14], but it should also be possible with interacting fields. The reason is that nonperturbative effects manifest themselves by the appearance of operator-valued zero modes of the field operators which do not give rise to additional singularities [7]. This will be the subject of further work.

Appendix A. The wave front set

The wave front set of a distribution is not only important for the (canonical) restriction of distributions but plays also a central role in the so-called microlocal analysis [8]. The definition starts by firstly considering only distributions with compact support. Let $\phi \in \mathcal{D}'(X)$, $X \subset \mathbb{R}^n$ open, be a distribution with compact support. A non-zero vector $k_0 \in \mathbb{R}^n$ is called a *regular direction* of ϕ if there exists an open conic neighbourhood V of k_0 such that the Fourier transform of ϕ is rapidly decreasing on V, i.e. $\sup_{k\in V}(1+|k|)^N|\hat{\phi}(k)| < \infty$ for all $N = 0, 1, 2, \ldots$; otherwise k_0 is called a *singular direction* of ϕ. The set of all singular directions of ϕ is denoted by $\Sigma(\phi)$. If ϕ has not necessarily compact support one considers localizations of ϕ, i.e., distributions of the form $h\phi$ where $h \in \mathcal{D}(X)$ is a smooth function with compact support which equals the constant function 1 in an open subset of X. Since each such $h\phi$ is a distribution with compact support, the set $\Sigma(h\phi)$ of all its singular directions is well defined. Now, if $x \in X$ is some point, one defines $\Sigma_x(\phi)$ as the intersection of all sets $\Sigma(h\phi)$ where h runs through the set of all functions in $\mathcal{D}(X)$ which are equal to the constant function 1 in an open neighbourhood of x. The wave front set $\mathrm{WF}(\phi)$ of a distribution $\phi \in \mathcal{D}'(X)$ is the set of all pairs (x, k) in $X \times (\mathbb{R}^n\backslash 0)$ such that $k \in \Sigma_x(\phi)$. There is an important connection, which we would like to mention, between the wave front set of a distribution $\phi \in \mathcal{D}'(X)$ and its singularities. A point $x \in X$ is a regular point of ϕ, i.e., ϕ is a smooth function in some neighbourhood of x if and only if $\Sigma_x(\phi)$ is the empty set; otherwise x is a singular point of Φ. Hence, if $x \in X$ is a singular point of ϕ then $\Sigma_x(\phi)$ is the set of all singular directions which are common to all localizations of ϕ at x and which 'cause' the singularity of ϕ at x.

Appendix B. The pullback operation

In this part of the appendix we introduce the pullback operation for distributions and outline its relation to the (canonical) restriction of distributions. Let $f : X \to Y$ denote a C^∞ mapping between open sets $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$. If $\phi : Y \to \mathbb{C}$ is a function on Y, the composite function $f^*\phi = \phi \circ f$ is a function on X and is called the pullback of ϕ via f; the operation f^*, which assigns to each function on Y the function $f^*\phi = \phi \circ f$ on X, is called the pullback operation. A central problem in distribution theory is to extend the pullback operation from smooth functions to distributions $\phi \in \mathcal{D}'(Y)$. Here also the wave front set plays an important role. The following theorem gives the right subspace of $\mathcal{D}'(Y)$ to which the pullback operation can be extended from the case of smooth functions.

Theorem B.1 ([8], theorem 8.2.4)**.** *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open subsets and let $f : X \to Y$ be a C^∞ map. Then the pullback $f^*\phi$ can be defined in one and only one way for all $\phi \in \mathcal{D}'(Y)$ with*

$$N_f \cap WF(\phi) = \emptyset \tag{B.1}$$

so that $f^*\phi = \phi \circ f$ *when* $\phi \in C^\infty(Y)$*. Hereby*

$$N_f = \{(f(x), \eta) \in Y \times \mathbb{R}^n : (d_x f)^t \eta = 0\}$$

and is called the set of normals of f.

From theorem 8.2.4 in [8] one also obtains

$$\mathrm{WF}(f^*\phi) \subset f^*\mathrm{WF}(\phi) \tag{B.2}$$

whenever $N_f \cap \mathrm{WF}(\phi) = \emptyset$, where

$$f^*\mathrm{WF}(\phi) = \{(x, (d_x f)^t \eta) : (f(x), \eta) \in \mathrm{WF}(\phi)\}.$$

Since the pullback operation is a (contravariant) function, i.e., $(g \circ f)^* = f^* \circ g^* (g : Y \to Z)$, one obtains from (B.2):

Corollary B.2. *Let* $f : X \to Y (X, Y \subset \mathbb{R}^m)$ *be a* C^∞ *diffeomorphism. Then*

$$WF(f^*\phi) = f^*WF(\phi)$$

for all $\phi \in \mathcal{D}'(Y)$*. (Note that* $N_f = Y \times \{0\}$*.)*

The relation to the (canonical) restriction of distributions goes as follows. Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets and $a \in U$. Further denote by $\iota : V \to U \times V$ the C^∞ mapping defined by $\iota(y) = (a, y)$. Then the restriction $\phi|_{x=a}$ equals the pullback $\iota^*\phi$. The restriction operation is thus a pullback operation with respect to a special $f (= \iota)$. If we compute the set of normals N_ι, we obtain $N_\iota = \{((a, y), (p, 0)) | y \in V, p \in \mathbb{R}^m\}$. Hence, condition (3.2) for the existence of the canonical restriction is equivalent to condition (B.1) for the existence of the pullback.

Remark B.3. If we denote by $\lambda : \mathbb{R}^3 \to \mathbb{R}^4$, $\tilde{\mathbf{x}} = (x^1, x^2, x^-) \mapsto (x^-/\sqrt{2}, x^1, x^2, -x^-/\sqrt{2})$ then $\lambda(\mathbb{R}^3) = \Sigma = \{x^0 + x^3 = 0\}$ and $\lambda = \kappa^{-1} \circ \tilde{\iota}_0$, where $\tilde{\iota}_0(\tilde{\mathbf{x}}) = (0, \tilde{\mathbf{x}})$. Hence

$$\lambda^*\phi = (\kappa^{-1} \circ \tilde{\iota}_0)^*\phi = \tilde{\iota}^*(\kappa_*\phi) = \phi|_\Sigma,$$

i.e., $\phi|_\Sigma$ is the pullback of ϕ with respect to λ. Note that λ is a smooth parametrization of Σ, but Σ has infinitely many. However, if μ is another smooth parametrization of Σ then $\lambda = \mu \circ (\mu^{-1} \circ \lambda)$, where $\mu^{-1} \circ \lambda$ is a C^∞ diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$. Hence, by corollary appendix B.2, $\lambda^*\phi$ exists if and only if $\mu^*\phi$ exists, and in this case $\lambda^*\phi$ and $\mu^*\phi$ differ only by multiplication of a smooth function—the determinant of the Jacobian of $\mu^{-1} \circ \lambda$.

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